Research Reports

Simultaneous Estimation of the Supply and Demand of Differentiated Audits: Evidence from the Municipal Audit Market

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1. Introduction

The purpose of this study is to analyze the supply and demand for municipal auditing services using simultaneous equations. Our approach differs from most prior studies which separately investigate either the supply-side determinants of audit fees or the demand-side determinants of auditor choice.1 We argue that ignoring the simultaneous

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endogeneity of audit fees and audit attributes engenders problems of parameter identification and estimation bias, which prevents meaningful interpretation of variable coefficients. For example, agency theory predicts that high-leverage/high-risk firms will use higher-quality auditors in an effort to reduce agency costs. Using single-equation demand models both Palmrose [1984] and Simunic and Stein [1987] find evidence which is inconsistent with this prediction. An explanation is that failing to control for the supply-side influence of client risk negates the demand-side effect. Supporting this explanation, we find that the predicted relation between client leverage/risk and the demand for a high-credibility auditor is observed only when supply-side influences are held constant.

The problem is formulated as follows. We assume that the audit is a differentiated product sold in a nonarbitrageable market. Simunic [1980] argues that the principal differentiating characteristic of the audit service is the identity of the supplier. Accordingly, we select auditor reputation, as conveyed by brand name, as the focus of our analysis.

The assumption that audits are traded in a nonarbitrageable market is equivalent to asserting that auditor reputation is indivisible. That is, low-reputation auditors cannot pool their credibility to offer the same reputation level (at the same cost) as that available from a single, high-reputation supplier. Rosen [1974] shows that when packages of product attributes cannot be "untied"—either because sellers cannot repackage existing products, or because they find it uneconomical to do so—the fee function for the product will be nonlinear. In our case, auditor reputation cannot be divided and repackaged, and therefore the fee function for auditor reputation is nonlinear. This recasts the standard textbook formulation of the analysis of prices and quantities (which assumes a homogeneous product with a constant unit price), to a marginal fee analysis. Marginal fees vary with the quantity of auditor reputation exchanged, even when the market is competitive.

Analysis of marginal fees and quantities transacted requires simultaneous estimation of supply and demand functions for auditor reputation. To identify the parameters of each equation, additional explanatory variables (supply and demand shifters that capture client and possibly auditor reputation) are needed.

2 Cross-sectional variation of audit attributes is also assumed by Simunic [1980], Dopuch and Simunic [1982], and Simunic and Stein [1987], and relates to earlier work on hedonic markets by Rosen [1974] and Lancaster [1966].

3 Simunic and Stein [1987] identify three audit service attributes along which differentiation might take place: (1) financial statement credibility, which is communicated by an audit firm's brand name; (2) contribution to the client's organizational control; and (3) the availability of management consulting services. Our analysis focuses on the first audit service attribute. Some support for this is found in O'Keefe, Simunic, and Stein [1994], who report that the intensity of nonaudit services has no relation to the quantity and mix of labor inputs utilized in the audit. They suggest that nonaudit activities can safely be ignored in empirical research involving audit fees.

4 A similar point is made by Benston [1979], who contends that the costs of coordination prevent a consortium of small auditing firms from assembling the same audit package at the same cost as that offered by a single, large supplier.
characteristics) are necessary. Empirical estimation of the simultaneous equations is complicated by the fact that marginal fees are not directly observable. We estimate marginal fees in two ways: first, by differentiating a fee function that is quadratic in auditor reputation and audit scope and, second, by replacing marginal fees with the log of total fees in the structural equations.

The sample consists of 162 U.S. cities with populations over 50,000 that were audited by external CPAs in 1985. Our main conclusion is that client characteristics (such as size, risk, and leverage) are important determinants of both the demand for auditor reputation and the marginal fee for auditor reputation. Thus, in order to obtain meaningful coefficients on demand variables in auditor choice models, supply-side effects must be taken into account. The converse is also true. We find (at least for the marginal fee analysis) that audit demand is an important determinant of municipal audit fees.

Section 2 discusses the econometric considerations involved in the simultaneous estimate of supply and demand equations for auditor reputation, highlighting the issue of the unobservability of marginal fees. Section 3 applies the estimation technique to the sample of municipal audits. Section 4 summarizes the paper and discusses its implications.

2. The Model

Consider the following two-equation system that in equilibrium jointly determines the values of each marginal fee $f'(R)$ and audit $R$ traded in the market.

$$
R = \beta_0 + \beta_1 f'(R) + x'\beta + \epsilon_d \quad \text{(demand)}
$$

$$
f'(R) = \gamma_0 + \gamma_1 R + z'\gamma + \epsilon_s \quad \text{(supply)}.
$$

In the demand equation, clients have exogenous characteristics (captured by the vector $x$) that create cross-client variations in the demand for auditor reputation. These characteristics, or demand shifters, measure the level of agency costs faced by the client. As discussed in section 3, we use the logarithm of the population of the municipality (size), total per capita debt issued in the fiscal year following the audit (leverage), and other agency-inspired measures as demand shifters for our sample of municipal audits. The parameter vector $\beta$ is common to all clients.

Interpretation of the demand equation is straightforward. In the case of a homogeneous product, the fee function $f(R)$ is linear in $R$, and the marginal fee function $f'(R)$ is a constant price per unit of reputation. This produces a standard demand curve that is downward sloping in price, with demand shifters represented by $x'\beta$. However, under our assumption that the audit is a differentiated product, $f(R)$ is nonlinear in $R$, and marginal fees vary. Nevertheless, we assume that $f'(R) > 0$, preserving the intuition that marginal reputation is positively priced. Also, the coefficient $\beta_1$ is expected to be negative, indicating that individual demands for auditor reputation are downward sloping in marginal fees.
In the supply equation, audit firms have exogenous characteristics (captured by the vector $z$) that create variations in their cost functions. For example, Simunic and Stein [1987] argue that auditors invest in technology, physical facilities, personnel and their knowledge, organizational control systems, etc., to efficiently produce a given credibility level. Additionally, client characteristics, such as size and complexity of operations, are inputs to the auditor's cost function. Taken together, these auditor and client characteristics enter the supply function for auditor reputation as *supply shifters*. We use the logarithm of the population of the municipality (client size), an index of municipal services (client complexity), and a qualitative measure indicating whether the municipality's fiscal year ends during the audit busy season as supply shifters for our sample of municipal audits. The parameter vector $\gamma$ is common to all auditors.

The interpretation of the supply equation is analogous to that of the demand equation; i.e., marginal fees $f'(R)$ are nonconstant and $f''(R) > 0$. Also, the coefficient $\gamma_1$ is expected to be positive, consistent with individual supply curves for auditor reputation that are upward sloping in marginal fees.

Estimation of the model requires marginal fees, which are not observable. If audit fees are observed and auditor reputation can be quantified, a marginal fee function can be estimated by taking the first derivative of a nonlinear fee function fit to sampled fee/reputation pairs (Rosen [1974]). A simple nonlinear fee function is $f(R) = \pi_0 + \pi_1 R + 1/2 \pi_2 R^2$. However, a quadratic fee function in conjunction with linear structural equations (such as (1)) results in an identification problem. Specifically, marginal fees are linear in $R$ (i.e., $f'(R) = \pi_1 + \pi_2 R$), making the dependent variables in the system perfectly correlated with one of their predictor variables (see Murray [1985]). To achieve parameter identification, the fee function must either have derivatives of degree two or higher, or contain additional predictor variables. The second condition applies in the first of our two empirical tests. We estimate marginal fees as the first derivative of a fee function that is quadratic in auditor reputation and audit scope, where audit scope is measured as the logarithm of revenues of the municipal client. Including the log of municipal revenues in the fee function achieves parameter identification because it is not one of the demand or supply shifters in the structural equations. Also, it seems reasonable to model audit fees not only as a function of auditor reputation but also as a function of the scope of the audit.

For our first analysis, we use principal components analysis to extract a continuous measure of reputation from three underlying variables: the number of SEC clients serviced by the auditor in 1985 ($NUMSEC$), combined revenues reported by the auditor's SEC clients in 1985 ($REVSEC$), and the number of sample municipalities serviced by the auditor in 1985.

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5 Derivation of a quadratic fee function for auditor reputation is given in Appendix A.
Information on the number of SEC clients serviced by the Big-Eight firms and the revenues reported for these clients is obtained from Who Audits America (Harris [1985]). We do not know the identities of the non-Big-Eight auditors in our sample. For these firms, we code the number of SEC clients as zero; the revenues reported for SEC clients as zero; and the number of municipalities audited as one. The goal of the principal components analysis is to isolate the underlying construct common to all measures, producing a continuous measure of reputation that contains less noise than the underlying variables. The first principal component of the three variables explains about 90% of their common variation. This factor is used as the reputation variable.

For our second analysis, we treat auditor reputation as a latent construct and, following prior research, use the Big-Eight/non-Big-Eight designation as its indicator variable. A nonlinear fee function cannot be fit to an indicator variable, but we observe that the assumption of a convex fee function implies positive correlations among marginal fees, total fees, and log fees. Thus, rather than deriving a marginal fee function, another approach is to replace marginal fees with total fees (or the log of total fees) in the structural equations and to use the Big-Eight/non-Big-Eight indicator variable as the measure of reputation. The disadvantage of this method is that replacing marginal fees with total or log fees biases the estimated coefficients in the structural equations. However, the magnitude of the bias is inversely related to the correlation between marginal fees and their replacement, approaching zero as the correlation approaches one.

To assess the reasonableness of this strategy, we note that a fee function that is quadratic in a normally distributed reputation variable exhibits correlation between fees and marginal fees equal to 

\[ \rho \left[ f(R), f'(R) \right] = \left( 1 + \frac{1}{2} CV^2 \left[ f'(R) \right] \right)^{-\frac{1}{2}}, \]

where \( CV[f'(R)] \) is the coefficient of variation of marginal fees (see Appendix B). Even a conservative estimate of the variation in marginal fees (such as \( CV = 0.5 \)) yields a high correlation (\( \rho > .9 \)) between marginal fees and total fees. An even more compelling argument for substituting total fees for marginal fees can be made when the fee function is exponential in auditor reputation. In this case, the correlation between total fees and marginal fees is always one. For certain functional forms, marginal fees are related to both log fees and total fees. We conservatively estimate that our classification of non-Big-Eight auditors as non-SEC service providers results in at most ten misclassifications of municipalities, with a probable margin of measurement error in \( NUMSEC \) and \( REVSEC \) of no more than 1% of the average values exhibited by Big-Eight firms. Empirically, our results are robust to other plausible assumptions concerning the SEC clientele of non-Big-Eight auditors in our sample.

For example, if we use hourly billing rates for staff auditors as a measure of marginal fees and assume that hourly billing rates are normally distributed with a centered, six standard deviation range of $20–$100, the coefficient of variation is approximately 0.13. This implies that the correlation between total fees and marginal fees is 0.994. A higher estimate, such as \( CV = 0.50 \), yields a correlation of 0.943.
fees. Ultimately, it is an empirical issue whether the data favor total fees or log fees as the preferred replacement for marginal fees.

The results of the simultaneous estimation of the supply and demand for auditor reputation in the municipal audit market are presented in the next section. Two sets of analyses are presented. In the first analysis, we estimate marginal fees using a fee function that is quadratic in auditor reputation and audit scope. We then input estimated marginal fees into the structural equations. In the second analysis, we replace marginal fees with log fees in the structural equations and use the Big-Eight/non-Big-Eight designation as an indicator of reputation. We use log fees, rather than total fees, to be consistent with previous research, and also because our empirical tests indicate the logarithmic form provides a better fit to the data.

Both approaches introduce error. While our continuous measure of auditor reputation captures both variation among the Big-Eight firms and between the Big-Eight and non-Big-Eight firms, it ignores possible variation among the non-Big-Eight firms. To the extent marginal fees and log fees are not perfectly correlated, the log fee analysis results in biased coefficient estimates in the structural equations. Despite these sources of error, both sets of results suggest significant insights from recasting the estimation problem from a single-equation to a simultaneous-equation framework.

3. Results

We focus on the municipal audit market because previous research (Zimmerman [1977], Baber [1983], and Evans and Patton [1987]) suggests the presence of a variety of signaling and monitoring incentives which are likely to affect the demand for auditing in the public sector. We specify the simultaneous equations model as:

\[
\text{REPUTATION} = \beta_0 + \beta_1 \text{MARGFEE} + \beta_2 \text{TAXSHARE} + \beta_3 \text{PROP-TAX} \\
+ \beta_4 \text{NEWDEBT} + \beta_5 \text{RATING} + \beta_6 \text{MANAGER} + \beta_7 \text{LN(POPUL)} \\
+ \beta_8 \text{MAYORTURN} + \beta_9 \text{B8OFFICE} + \epsilon_d \\
\text{(demand)}
\]

\[
\text{MARGFEE} = \gamma_0 + \gamma_1 \text{REPUTATION} + \gamma_2 \text{LN(POPUL)} + \gamma_3 \text{SERVICES} \\
+ \gamma_4 \text{DEBT} + \gamma_5 \text{RATING} + \gamma_6 \text{SEASON} + \epsilon_s \\
\text{(supply)}
\]

We present two sets of results. In our first set of tests (reported in section 3.3.1), REPUTATION is a continuous variable, measured as the factor

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8 An example is the case where fees are a double exponential function of a log normally distributed reputation variable.

9 Although the correlation between marginal fees and log fees cannot be derived analytically, simulations based on a quadratic fee function with \( CV(f'(R)) = 0.15 \) yield a correlation exceeding 0.9. For our data, the correlation between estimated marginal fees (from the first analysis) and log fees is 0.66. The correlation between estimated marginal fees and total fees is 0.68.
score for the auditor obtained from the principal components analysis of the underlying variables \textit{NUMSEC}, \textit{REVSEC}, and \textit{NUMCITY}. \textit{MARGFEE} is the marginal fee for the audit, measured as the first derivative of a nonlinear fee function given by \( \textit{FEE} = \gamma_1 \textit{REPUTATION} + \gamma_2 \textit{LN} (\textit{REV}) + \gamma_3 \textit{REPUTATION} \times \textit{LN} (\textit{REV}) + \gamma_4 \textit{REPUTATION}^2 + \gamma_5 \left[ \textit{LN} (\textit{REV}) \right]^2 + \epsilon_5 \), where \( \textit{LN} (\textit{REV}) \) is the logarithm of total revenues of the municipal client.\(^{10}\) In our second set of tests (reported in section 3.3.2), the continuous variable \textit{REPUTATION} is replaced by a dichotomous variable \textit{(BIG-EIGHT)} and \textit{MARGFEE} is replaced by the logarithm of the fee charged for the audit \( [\textit{LN} (\textit{FEE})] \).

In both sets of tests, demand shifters consist of \textit{TAXSHARE}, total per capita revenues excluding those received from other governments; \textit{PROP-TAX}, the percentage of local revenues obtained from property taxes; \textit{NEWDEBT}, total per capita debt issued in the fiscal year following the audit; \textit{RATING}, a discrete variable indicating the number of categories below AAA a municipality's general obligation bonds are rated; \textit{MANAGER}, a qualitative variable indicating whether the municipality uses a city-manager form of government; \( \textit{LN} (\textit{POPUL}) \), the logarithm of the population of the municipality; \textit{MAYORTURN}, a qualitative variable indicating whether the mayor failed to be reelected in the subsequent election; and \textit{B8OFFICE}, a qualitative variable indicating whether any Big-Eight firms have offices located in the city. Supply shifters consist of \( \textit{LN} (\textit{POPUL}) \); \textit{SERVICES}, an index of municipal services; \textit{DEBT}, total per capita long-term debt; \textit{RATING}, and \textit{SEASON}, a qualitative variable indicating whether a municipality's fiscal year ends during the audit busy season. Our choice of demand and supply shifters is motivated by prior evidence from the public sector audit literature. These variables are discussed below.

3.1 DEMAND SHIFTERS

Municipalities have exogenous characteristics (previously referred to as the vector \( \textit{x} \)) that create cross-sectional variations in the demand for auditor reputation. These characteristics, or demand shifters, measure the level of agency costs faced by the municipality. We posit that voters with a relatively high perceived tax burden are likely to reflect greater concern over the use of public funds. Politicians seeking continued support respond to this concern by providing auditors of higher credibility. \textit{TAXSHARE}, which reflects the actual contribution to the cost of public

\(^{10}\) Not surprisingly, the fee function is subject to a high degree of multicollinearity (variance inflation factors greater than 35). Thus, parameter estimates, while unbiased, are unstable. This parameter instability does not result in unstable fee predictions, but it does produce unstable (although unbiased) marginal fee predictions. As a result, we estimate marginal fees that are negative for some firms. Nevertheless, we find a significantly positive correlation between marginal fees and total fees (also log fees) for our data. This suggests that the noise in the estimates of marginal fees does not invalidate their use in the structural equations model.
goods by the average citizen, is therefore expected to have a positive coefficient in the demand equation. Taxpayers often systematically underestimate their tax share due to the complexity of the revenue system (Pommerehne and Schneider [1978]). At the local government level, property taxes are the most visible source of tax revenue since they are paid at infrequent intervals and in relatively large amounts. Higher values of PROP-TAX reflect more visible tax structures and are also expected to be positively related to the choice of a high-reputation audit firm.

Politicians have incentives to reduce interest costs in order to increase the resources available for other programs (Zimmerman [1977]). They also gain individual recognition as a result of improved municipal credit ratings (Ingram [1984]). Interest costs can be reduced, and credit ratings increased, by enhancing financial statement credibility through the selection of a brand-name (high-reputation) auditor (Simunic and Stein [1987]). The magnitude of potential interest savings, and therefore the demand for a high-reputation auditor, are expected to be positively related to the level of new debt issues in subsequent periods. Likewise, the political incentives for improvement in credit rating are expected to be greater when the existing credit risk is high. Thus, the coefficients on both NEWDEBT and RATING in the demand equation are expected to be positive.

Zimmerman [1977] finds that cities operated under a manager form of government are more likely to engage large national auditors. Baber [1983] and Evans and Patton [1987] argue that the demand for auditing increases with the number and magnitude of wealth transfers, measured by the size of the municipality. Thus, the coefficients on MANAGER and LN(POPUL) in the demand equation are expected to be positive. Baber [1983] argues that political incumbents have an incentive, increasing with the level of political competition, to demonstrate conformance with constituent interest groups' goals. The demand for a high-reputation auditor is therefore expected to increase with the level of political competition, as measured by MAYORTURN. Finally, the coefficient on B8OFFICE, a control variable indicating the presence of a Big-Eight firm within a city, is expected to be positive.

3.2 Supply Shifters

Auditors' cost functions are influenced by exogenous client and auditor characteristics, collectively referred to as supply shifters (vector z). The supply shifters in our model are largely client-specific. Baber, Brooks, and Ricks [1987] and Ward, Edler, and Kattelus [1994] find that local government audit fees are positively related to client size, complexity, and the magnitude or riskiness of outstanding debt. Thus, the coefficients on LN(POPUL), SERVICES, DEBT, and RATING in the supply equation are expected to be positive. Both studies also report a significantly positive relation between audit fees and audit firm size, implying a positive coefficient for reputation. Finally, audit fees are expected to increase dur-
ing peak load periods, which suggests that \textit{SEASON} will have a positive coefficient.

3.3 RESULTS

The sample consists of 162 U.S. cities with populations over 50,000 that were audited by external CPAs in 1985. By limiting the analysis to these municipalities, we obtain a sample in which the proportions of Big-Eight (52%) and non-Big-Eight (48%) audits are roughly equivalent, and in which the clients of Big-Eight and non-Big-Eight firms are of comparable size. Auditor identity, audit fee, form of government, and fiscal year are taken from a survey conducted in early 1986.\footnote{See Ingram and Robbins [1987] for a description of the survey instrument and tests for nonresponse bias, and Copley [1993] for further details on the source or measurement of specific variables.} General Obligation bond rating is obtained from Moody’s [1985] and mayoral turnover is coded using the \textit{Directory of City Policy Officials} (National League of Cities [1985; 1989]). All financial measures are obtained from the \textit{Survey of Governments: Annual Financial Statistics} (U.S. Bureau of Census [1985; 1986]).

3.3.1. Marginal Fee Analysis. Table 1, panel A presents the results of a simultaneous equations estimation of the demand equation using the continuous variable, \textit{REPUTATION}, and marginal fees derived from the quadratic fee function. Ordinary least squares results are also presented in panel B for comparative purposes. In panel A, we use two-stage least squares to obtain parameter estimates and estimates of the asymptotic standard errors. With the exception of \textit{MANAGER} and \textit{MAYORTURN}, all variables have coefficients of the predicted sign. However, only \textit{TAXSHARE}, \textit{PROP TAX}, \textit{LNPOPUL}, and \textit{MARGFM} are significant (\textit{p} values for the coefficients on these variables are all 0.0001, based on two-tailed tests). The results suggest that the demand for a high-reputation auditor is largely a function of municipal size and the perceived tax burden, and not of municipal risk/leverage, as would be predicted by agency theory (the coefficients on \textit{NEWDEBT} (\textit{p} = 0.27) and \textit{RATING} (\textit{p} = 0.46) are not significant at conventional levels). The significantly negative coefficient on the marginal audit fee (\textit{p} = 0.0001) indicates a downward-sloping demand for auditor reputation. The overall model is highly significant (F value = 27.68), with an adjusted $R^2$ of 0.62.

Table 2 presents the results of a simultaneous equations estimation of the supply equation for auditor reputation, again using the two-stage least squares procedure. As before, single-equation results are also presented for comparison. In the two-stage results (panel A), we observe that the client-specific supply shifters (\textit{LNPOPUL}, \textit{DEBT}, and \textit{RATING}) are significantly positively related to marginal audit fees (two-tailed \textit{p}-values are 0.0001 for \textit{LNPOPUL} and 0.04 for \textit{DEBT} and \textit{RATING}). However, the auditor-specific supply shifter (\textit{SEASON}) is not significant at conventional levels (\textit{p} = 0.16). Also, the coefficient on \textit{REPUTATION} is marginally
TABLE 1
Parameter Estimates for the Demand Function for Auditor Reputation (REPUTATION) for the Fiscal Year Ended in 1985 for a Sample of 162 City Governments with Populations Greater than 50,000a

Panel A: Two-Stage Least Squares

Predictor Variables | Expected Sign | Coefficient | Asymptotic Std. Errorb | Z-Statistic | P-Value (two-tail)
--- | --- | --- | --- | --- | ---
Intercept | | -25.755 | 2.573 | -10.011 | 0.0001
TAXSHAREc | + | 2.034 | 0.200 | 10.150 | 0.0001
PROP-TAXd | + | 1.150 | 0.188 | 6.022 | 0.0001
NEWDEBTc | + | 0.080 | 0.073 | 1.105 | 0.2711
RATINGf | + | 0.016 | 0.021 | 0.744 | 0.4583
MANAGERg | + | -0.045 | 0.060 | -0.757 | 0.4500
LN(POPUL)h | + | 2.012 | 0.203 | 9.915 | 0.0001
MAYORTURNi | + | -0.045 | 0.061 | -0.749 | 0.4552
B8OFFICEj | + | 0.064 | 0.067 | 0.954 | 0.3417
MARGFEE | - | -0.202 | 0.023 | -8.894 | 0.0001

Model F-statistic (p-value): 27.684 (0.0001) Adjusted $R^2$: 0.621

Panel B: Ordinary Least Squares (for Comparison Purposes)

Predictor Variables | Expected Sign | Coefficient | Std. Error | T-Statistic | P-Value (two-tail)
--- | --- | --- | --- | --- | ---
Intercept | | -3.231 | 1.261 | -2.562 | 0.0114
TAXSHAREc | + | 0.376 | 0.205 | 1.834 | 0.0686
PROP-TAXd | + | 0.846 | 0.515 | 1.642 | 0.1027
NEWDEBTc | + | 0.049 | 0.202 | 0.241 | 0.8099
RATINGf | + | -0.093 | 0.049 | -1.919 | 0.0568
MANAGERg | + | 0.118 | 0.159 | 0.746 | 0.4569
LN(POPUL)h | + | 0.241 | 0.109 | 2.214 | 0.0283
MAYORTURNi | + | 0.160 | 0.156 | 1.028 | 0.3058
B8OFFICEj | + | -0.007 | 0.185 | -0.040 | 0.9681

Model F-statistic (p-value): 2.736 (0.0076) Adjusted $R^2$: 0.079

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*aThe dependent variable, REPUTATION, is the first principal component of three continuous measures of auditor reputation: (1) the number of SEC clients serviced by the auditor in 1985 (NUMSEC), (2) combined revenues reported by the auditor's SEC clients in 1985 (REVSEC), and (3) the number of sample municipalities serviced by the auditor in 1985 (NUMCITY). Information on SEC clients serviced by the Big-Eight firms is obtained from Who Audits America (Harris [1985]). Values for NUMSEC, REVSEC, and NUMCITY for non-Big-Eight firms are set to zero, zero, and one respectively.

bThe asymptotic standard errors of the coefficient estimates are for two-stage least squares, using the 2SLS option provided under the SYSLIN procedure of the SAS/ETS software package.

cTotal per capita revenues excluding those received from other governments.

dThe percentage of local revenues obtained from property taxes.

eTotal per capita debt issued during fiscal year 1986.

fA discrete variable indicating the number of categories below AAA a municipality's general obligation bonds are rated.

gA qualitative variable indicating whether the municipality uses a city manager form of government.

hLogarithm of the population of the municipality (1980 census).

iA qualitative variable indicating whether the mayor failed to be reelected in the subsequent election.

jA qualitative variable indicating whether any Big-Eight firms have offices located in the municipality.

kThe value of the marginal fee (MARGFEE) predicted by the first-stage, reduced form regression under 2SLS. The marginal fee is derived as the first derivative of a quadratic fee function fit to REPUTATION and the logarithm of city revenues, LN(REV).

significant ($p = 0.08$) and positive, which indicates that the supply function for auditor reputation is upward sloping in marginal fees. Again, the overall model is highly significant ($F$ value = 22.65), with an adjusted $R^2$ of 0.45.
### Table 2
Parameter Estimates for the Supply Function for Auditor Reputation for the Fiscal Year Ended in 1985 for a Sample of 162 City Governments with Populations Greater than 50,000

#### Panel A: Two-State Least Squares

<table>
<thead>
<tr>
<th>Predictor Variables</th>
<th>Expected Sign</th>
<th>Coefficient</th>
<th>Asymptotic Std. Error</th>
<th>Z-Statistic</th>
<th>P-Value (two-tail)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>-92.465</td>
<td>16.721</td>
<td>-5.530</td>
<td>0.0001</td>
</tr>
<tr>
<td>LN(POPUL)^c</td>
<td>+</td>
<td>6.589</td>
<td>1.630</td>
<td>4.043</td>
<td>0.0001</td>
</tr>
<tr>
<td>SERVICES^d</td>
<td>+</td>
<td>1.560</td>
<td>0.668</td>
<td>2.333</td>
<td>0.2090</td>
</tr>
<tr>
<td>DEBT^e</td>
<td>+</td>
<td>1.968</td>
<td>0.946</td>
<td>2.080</td>
<td>0.0892</td>
</tr>
<tr>
<td>RATING^f</td>
<td>+</td>
<td>1.337</td>
<td>0.648</td>
<td>2.062</td>
<td>0.0409</td>
</tr>
<tr>
<td>SEASON^g</td>
<td>+</td>
<td>3.400</td>
<td>2.406</td>
<td>1.413</td>
<td>0.1596</td>
</tr>
<tr>
<td>REPUTATION (predicted)</td>
<td></td>
<td>6.586</td>
<td>3.800</td>
<td>1.773</td>
<td>0.0851</td>
</tr>
</tbody>
</table>

Model F-statistic (p-value): 22.653 (0.0001) Adjusted $R^2$: 0.447

#### Panel B: Ordinary Least Squares (for Comparison Purposes)

<table>
<thead>
<tr>
<th>Predictor Variables</th>
<th>Expected Sign</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>T-Statistic</th>
<th>P-Value (two-tail)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>-4.275</td>
<td>0.596</td>
<td>-7.177</td>
<td>0.0001</td>
</tr>
<tr>
<td>LN(POPUL)^c</td>
<td>+</td>
<td>0.641</td>
<td>0.054</td>
<td>11.861</td>
<td>0.0001</td>
</tr>
<tr>
<td>SERVICES^d</td>
<td>+</td>
<td>0.045</td>
<td>0.031</td>
<td>1.459</td>
<td>0.1467</td>
</tr>
<tr>
<td>DEBT^e</td>
<td>+</td>
<td>0.108</td>
<td>0.048</td>
<td>2.261</td>
<td>0.0251</td>
</tr>
<tr>
<td>RATING^f</td>
<td>+</td>
<td>0.118</td>
<td>0.026</td>
<td>4.499</td>
<td>0.0001</td>
</tr>
<tr>
<td>SEASON^g</td>
<td>+</td>
<td>0.050</td>
<td>0.091</td>
<td>0.546</td>
<td>0.5858</td>
</tr>
</tbody>
</table>

Model F-statistic (p-value): 50.485 (0.0001) Adjusted $R^2$: 0.606

*aThe dependent variable, marginal fees (MARGFEE), is the value of the first derivative of a quadratic hedonic fee function fit to REPUTATION and the logarithm of city revenues, LN(REV).

*bThe asymptotic standard errors of the coefficient estimates are for the two-stage least squares, using the 2SLS option provided under the SYSLIN procedure of the SAS/ETS software package.

*cThe logarithm of the size of the municipality measured by population (1980 census).

*dAn index of municipal services.

*eTotal per capita long-term debt outstanding at year-end (1985).

*fA discrete variable indicating the number of categories below AAA a municipality's general obligation bonds are rated.

*gA qualitative variable indicating whether the municipality's fiscal year-end falls between October and February.

*hThe value predicted by the first-stage, reduced form model for auditor reputation (REPUTATION). REPUTATION is the first principal component of three continuous measures of auditor reputation: (1) the number of SEC clients serviced by the auditor in 1985 (NUMSEC), (2) combined revenues reported by the auditor's SEC clients in 1985 (REVSEC), and (3) the number of sample municipalities serviced by the auditor in 1985 (NUMCITY). Information on SEC clients serviced by the Big-Eight firms is obtained from Who Audits America (Harris [1985]). Values for NUMSEC, REVSEC, and NUMCITY for non-Big-Eight firms are set to zero, zero, and one respectively.

Of particular interest in table 2, panel A are the significantly positive ($p = 0.04$) coefficients on the risk and leverage measures, RATING and DEBT. Taken together, the results in tables 1 and 2 for the simultaneous equations estimation suggest that although client leverage and risk measures do not significantly influence the demand for auditor reputation, they do influence audit fees. An implication is that ignoring fee effects can lead to an erroneous conclusion that client leverage/risk and reputation are negatively related. To test this implication, we delete the marginal fees from the demand equation and conduct a single equation OLS analysis on the remaining variables. The results are presented in panel B of table 1. Similar to the results in panel A, the
coefficients on $LN(POPUL)$ and $TAXSHARE$ are significantly positive (two-tailed $p$-values are 0.03 for $LN(POPUL)$ and 0.07 for $TAXSHARE$). The coefficient on $PROP-TAX$ is also positive and significant on the 0.10 level. However, the coefficient on $RATING$ is significantly negative ($p = 0.06$). This result underscores the importance of controlling for supply-side influences when testing for demand effects.

3.3.2. Log Fee Analysis. Table 3 presents the results of our second estimation of the demand equation. In this analysis, we use the dichotomous variable, $BIG-EIGHT$, rather than the continuous variable $REPUTATION$, as an indicator of auditor reputation and use log fees, rather than marginal fees. Following Copley, Doucet, and Gaver [1994], we use two-stage generalized least squares (Amemiya [1978]) to obtain parameter estimates and estimates of the asymptotic standard errors. Unlike the marginal fee analysis, in the log fee analysis the coefficients on $NEWDEBT$ ($p = 0.06$) and $RATING$ ($p = 0.06$) are significantly positive, as predicted by agency theory. The coefficients on $TAXSHARE$ ($p = 0.01$), $PROP-TAX$ ($p = 0.05$), $LN(POPUL)$ ($p = 0.01$), and $BOFFICE$ ($p = 0.04$) are also all significantly positive, as predicted. Similar to the marginal fee analysis, the coefficients on $MANAGER$ ($p = 0.73$) and $MAYORTURN$ ($p = 0.16$) are not significant at conventional levels. The significantly negative coefficient on audit fee ($p = 0.02$) indicates a downward-sloping demand for auditor reputation. The overall model is highly significant (chi-square = 38.88, with 9 d.f.), with a pseudo $R^2 = 0.18$.13

In panel B, we delete the fee variable from the demand equation and conduct a single-equation probit analysis on the remaining variables. As in panel A, the coefficients on $TAXSHARE$ ($p = 0.06$), $PROP-TAX$ ($p = 0.06$), and $LN(POPUL)$ ($p = 0.02$) are significantly positive, but the coefficient on $RATING$ (which reflects the riskiness of municipal debt; $p = 0.07$) changes sign. These results, in conjunction with the findings from the marginal fee analysis, suggest the following interpretation. High-risk clients must pay (incrementally) more for high-reputation auditors. When these fee effects are ignored, as in the single-equation analysis (panel B), the data suggest, erroneously, the high-risk clients demand low-reputation auditors. However, when fee effects are taken into account, as in panel A, there is a positive relation between client-specific risk and the demand for auditor reputation. Feltham, Hughes, and Simunic [1991] suggest that the failure of their tests to yield strong support for a hypothesized positive relation between client risk and audit demand in the new issues market is due to the inability to control for such supply-side effects. Our results confirm this suggestion.

12 Comparable results are obtained using an alternative two-stage procedure (Maddala [1983, pp. 242–45]). Application of three-stage least squares (or iterated three-stage least squares) did not qualitatively change the results. Ordinary least squares estimation is inappropriate in this setting because $REPUTATION$ and $LN(FEE)$ are jointly determined endogenous variables.

13 McFadden [1979] states that a pseudo $R^2$ in excess of 0.20 indicates an excellent fit.
### Table 3

Parameter Estimates for the Demand Function for Auditor Reputation for the Fiscal Year Ended 1985 for a Sample of 162 City Governments with Populations Greater than 50,000

#### Panel A: Simultaneous Equations

<table>
<thead>
<tr>
<th>Predictor Variables</th>
<th>Expected Sign</th>
<th>Coefficient</th>
<th>Asymptotic Std. Error</th>
<th>Z-Statistic</th>
<th>P-Value (two-tail)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>-28.584</td>
<td>10.333</td>
<td>-2.747</td>
<td>0.006</td>
</tr>
<tr>
<td>TAXSHARE</td>
<td>+</td>
<td>2.231</td>
<td>0.860</td>
<td>2.596</td>
<td>0.009</td>
</tr>
<tr>
<td>PROP-TAX</td>
<td>+</td>
<td>2.471</td>
<td>1.279</td>
<td>1.932</td>
<td>0.053</td>
</tr>
<tr>
<td>NEWDEBT</td>
<td>+</td>
<td>1.355</td>
<td>0.709</td>
<td>1.910</td>
<td>0.056</td>
</tr>
<tr>
<td>RATING</td>
<td>+</td>
<td>0.646</td>
<td>0.347</td>
<td>1.862</td>
<td>0.063</td>
</tr>
<tr>
<td>MANAGERS</td>
<td>+</td>
<td>-0.130</td>
<td>0.383</td>
<td>-0.338</td>
<td>0.735</td>
</tr>
<tr>
<td>LN(POPUL)</td>
<td>+</td>
<td>4.098</td>
<td>1.606</td>
<td>2.551</td>
<td>0.011</td>
</tr>
<tr>
<td>MAYORTURN</td>
<td>+</td>
<td>0.537</td>
<td>0.384</td>
<td>1.396</td>
<td>0.163</td>
</tr>
<tr>
<td>B8OFFICE</td>
<td>+</td>
<td>1.597</td>
<td>0.797</td>
<td>2.005</td>
<td>0.045</td>
</tr>
<tr>
<td>LN(FEE)</td>
<td>-</td>
<td>-6.443</td>
<td>2.760</td>
<td>-2.334</td>
<td>0.019</td>
</tr>
</tbody>
</table>

(Predicted)

Model $\chi^2$: 58.860 (9 d.f.); $p$-value: 0.001. Pseudo $R^2$: 0.18

#### Panel B: Single Equation Logit (for Comparison Purposes)

<table>
<thead>
<tr>
<th>Predictor Variables</th>
<th>Expected Sign</th>
<th>Coefficient</th>
<th>Asymptotic Std. Error</th>
<th>$T$-Statistic</th>
<th>P-Value (two-tail)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>-5.030</td>
<td>1.886</td>
<td>-2.667</td>
<td>0.008</td>
</tr>
<tr>
<td>TAXSHARE</td>
<td>+</td>
<td>0.554</td>
<td>0.293</td>
<td>1.886</td>
<td>0.059</td>
</tr>
<tr>
<td>PROP-TAX</td>
<td>+</td>
<td>1.403</td>
<td>0.737</td>
<td>1.905</td>
<td>0.057</td>
</tr>
<tr>
<td>NEWDEBT</td>
<td>+</td>
<td>0.092</td>
<td>0.289</td>
<td>0.317</td>
<td>0.751</td>
</tr>
<tr>
<td>RATING</td>
<td>+</td>
<td>-0.121</td>
<td>0.068</td>
<td>-1.788</td>
<td>0.074</td>
</tr>
<tr>
<td>MANAGERS</td>
<td>+</td>
<td>0.108</td>
<td>0.218</td>
<td>0.495</td>
<td>0.620</td>
</tr>
<tr>
<td>LN(POPUL)</td>
<td>+</td>
<td>0.377</td>
<td>0.160</td>
<td>2.354</td>
<td>0.018</td>
</tr>
<tr>
<td>MAYORTURN</td>
<td>+</td>
<td>0.218</td>
<td>0.214</td>
<td>1.016</td>
<td>0.310</td>
</tr>
<tr>
<td>B8OFFICE</td>
<td>+</td>
<td>0.034</td>
<td>0.251</td>
<td>0.134</td>
<td>0.893</td>
</tr>
</tbody>
</table>

Model $\chi^2$: 24.225 (8 d.f.); $p$-value: 0.0021 Pseudo $R^2$: 0.11

---

The dependent variable, BIG-EIGHT, takes on the value one of the auditor of the selected municipality is one of the eight largest accounting firms (n = 84), and is zero otherwise (n = 78).

The simultaneous equation generalized probit model, the two-stage generalized least squares estimation procedure, and the asymptotic standard errors of the coefficient estimates are described in Amemiya [1978].

Total per capita revenues excluding those received from other governments.

The percentage of local revenues obtained from property taxes.

Total per capita debt issued during fiscal year 1986.

A discrete variable indicating the number of categories below AAA a municipality's general obligation bonds are rated.

A qualitative variable indicating whether the municipality uses a city manager form of government.

Logarithm of the population of the municipality (1980 census).

A qualitative variable indicating whether the mayor failed to be reelected in the subsequent election.

A qualitative variable indicating whether any Big-Eight firms have offices located in the municipality.

The value of $\ln(\text{FEE})$ predicted by the first-stage, reduced form regression. FEE is the fee charged by the auditor for the audit engagement.

Table 4 presents the results of a simultaneous equations estimation of the supply equation for auditor reputation. The most significant determinants of audit fees are the size of the municipality, $\ln(\text{POPUL})$ ($p = 0.0001$), and the riskiness of municipal debt, $\text{RATING}$ ($p = 0.001$). This result reinforces the conclusion that client riskiness influences both audit fees and audit demand. However, unlike the supply equation...
TABLE 4
Simultaneous Equations Parameter Estimates for the Supply Function for Auditor Reputation for the Fiscal Year Ended in 1985 for a Sample of 162 City Governments with Populations Greater than 50,000*

<table>
<thead>
<tr>
<th>Predictor Variables</th>
<th>Expected Sign</th>
<th>Coefficient</th>
<th>Asymptotic Std. Error</th>
<th>Z-Statistic</th>
<th>P-Value (two-tail)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td></td>
<td>-3.204</td>
<td>1.246</td>
<td>-2.572</td>
<td>0.010</td>
</tr>
<tr>
<td>LN(POPUL)</td>
<td>+</td>
<td>0.531</td>
<td>0.121</td>
<td>4.387</td>
<td>0.000</td>
</tr>
<tr>
<td>SERVICES</td>
<td>+</td>
<td>0.072</td>
<td>0.050</td>
<td>1.448</td>
<td>0.147</td>
</tr>
<tr>
<td>DEBT</td>
<td>+</td>
<td>0.113</td>
<td>0.069</td>
<td>1.637</td>
<td>0.102</td>
</tr>
<tr>
<td>RATING</td>
<td>+</td>
<td>0.143</td>
<td>0.043</td>
<td>3.336</td>
<td>0.001</td>
</tr>
<tr>
<td>SEASON</td>
<td>+</td>
<td>0.167</td>
<td>0.162</td>
<td>1.031</td>
<td>0.303</td>
</tr>
<tr>
<td>BIG-EIGHT (predicted)</td>
<td></td>
<td>0.178</td>
<td>0.149</td>
<td>1.193</td>
<td>0.233</td>
</tr>
</tbody>
</table>

Model F-statistic (p-value): 43.114 (0.0001)  Adjusted $R^2$: 0.611

*The dependent variable, LN(FEE), is the logarithm of the audit fee for the municipality in 1985. The simultaneous equation generalized probit model, the two-stage generalized least squares estimation procedure, and the asymptotic standard errors of the coefficient estimates are described in Amemiya [1978].

1Logarithm of the size of the municipality measured by population (1980 census).
2An index of municipal services.
3Total per capita long-term debt outstanding at year-end (1985).
4A discrete variable indicating the number of categories below AAA a municipality's general obligation bonds are rated.
5A qualitative variable indicating whether the municipality's fiscal year-end falls between October and February.
6The value predicted by the first stage, reduced form probit model of auditor choice, where BIG-EIGHT = 1 if the auditor of the selected municipality is one of the eight largest firms (n = 84), and is zero otherwise (n = 78).

results reported in table 2 (using marginal fees), the reputation variable (BIG-EIGHT) is not significant ($p = 0.23$). Thus, while these results indicate the importance of controlling for supply-side influences when investigating demand effects (table 3), they provide little evidence that audit demand is an important determinant of municipal audit fees. The overall supply model is highly significant ($F$ value = 43.11), with an adjusted $R^2$ of 0.61.

4. Summary and Discussion

We estimate the supply and demand for auditor reputation using data from the municipal audit market, under the assumption that the audit is a differentiated product exchanged in a nonarbitrageable market. Hence, the fee function is nonlinear in reputation and marginal fees are nonconstant. We address the unobservability of marginal fees in two ways. First, we estimate marginal fees by differentiating a fee function that is quadratic in auditor reputation and audit scope. Second, we replace marginal fees with the log of total fees in the structural equations. Both approaches introduce their own types of error: the marginal fee analysis suffers from the potentially noisy measurement of auditor reputation, and (to the extent that marginal fees and log total fees are not perfectly correlated) the log fee analysis can result in biased coefficient estimates in the structural equations.
Our main conclusion is that client characteristics (such as size, risk, and leverage) are important determinants of both the demand for auditor reputation and the marginal fee for auditor reputation. Thus, in order to obtain meaningful coefficients on demand variables in auditor choice models, supply-side effects must be taken into account. For our sample (and also that of Feltham, Hughes, and Simunic [1991]), this effect is most apparent in the relation between audit demand and client leverage/risk. The converse is also true. We find (at least in our marginal fee analysis) that audit demand is an important determinant of municipal audit fees.

APPENDIX A

Derivation of the Fee Function for Auditor Reputation

The client chooses an auditor of reputation \( R \) in combination with nonaudit resources \( N \) (acquired at price 1) to maximize net expected firm value \( V(R, N; \xi) = G(R, N; \xi) - f(R) - N \), where \( G(R, N; \xi) \) is a strictly concave gross valuation function, \( f(R) \) is the exogenous fee function, and \( N \) is the cost of the numeraire input. The parameter \( \xi \) captures client characteristics \( x \) via an index function \( \xi = x' \beta \) where \( \beta \) is common to all clients in the market. Client characteristics are distributed normally, with mean \( \mu_x \) and variance \( \Sigma_x \). The client combines auditor reputation \( R \) and nonaudit resources \( N \) to produce gross firm value via the production function:

\[
G(R, N; \xi) = \frac{\theta}{2} (R - \xi)^2 + N, \quad (1)
\]

where \( \theta > 0 \) and is common to all clients.

Analogously, the auditor selects the level of reputation \( R \) to maximize profit \( f(R) - C(R; \xi) \), where \( C(R; \xi) \) is the auditor’s cost function which takes the form:

\[
C(R; \xi) = \alpha_0 + \alpha_1 (R - \xi) + \frac{\alpha_2}{2} (R - \xi)^2, \quad \alpha_2 > 0. \quad (2)
\]

Here, \( \xi = z' \gamma \) is an index of auditor attributes differentiating auditors in the market, and \( \gamma \) is common to all auditors. Auditor attributes are distributed normally, with mean \( \mu_z \) and variance \( \Sigma_z \).

These assumptions lead to the following implicit demand and supply functions for auditor reputation:

\[
R = -\frac{1}{\theta} f'(R) + x' \beta. \quad \text{(Implicit demand function.)} \quad (3)
\]

\[
R = \frac{1}{\alpha_2} f'(R) + z' \gamma - \frac{\alpha_1}{\alpha_2}. \quad \text{(Implicit supply function.)} \quad (4)
\]

**Proposition.** The fee function \( f(R) \) for auditor reputation that clears the market is given by:

\[
f(R) = \pi_0 + \pi_1 R + \frac{1}{2} \pi_2 R^2 \quad (5)
\]
where:
\[ \pi_0 = \text{indeterminate} \]
\[ \pi_1 = \frac{\alpha_1 \theta + \theta \alpha_2 \mu_x^\gamma - \theta \alpha_2 \mu_z^\gamma}{[\alpha_2 + \theta]} \]
\[ \pi_2 = \frac{\alpha_2 [\theta \mu_x^\gamma + \alpha_2 \mu_z^\gamma - \alpha_1] [\theta \gamma^\gamma - (\gamma \gamma)]}{[\alpha_2 + \theta] [\theta \gamma^\gamma + \alpha_2 \gamma^\gamma]} \]

From (5), we have the marginal fee \( f'(R) = \pi_1 + \pi_2 R \). Substituting \( f'(R) \) into the implicit demand and supply functions for auditor reputation (3) and (4), and rearranging terms, yields expressions for the optimal client demand and auditor supply functions:

\[ R = -\frac{\pi_1}{\theta + \pi_2} + \frac{\theta}{\theta + \pi_2} x^\beta \]

[optimal demand, given client attributes \( x \)]

\[ R = -\frac{\pi_1 - \alpha_1}{\pi_2 - \alpha_2} - \frac{\alpha_2}{\pi_2 - \alpha_2} z^\gamma \]

[optimal supply, given auditor attributes \( z \)].

Given the assumed distribution for \( x \) and \( z \), the distributions of optimal individual demand and supply schedules in the market are:

\[ R \sim N\left[ \frac{\pi_1}{\theta + \pi_2} + \frac{\theta}{\theta + \pi_2} \mu_x^\beta, \frac{\theta^2}{(\theta + \pi_2)^2} \gamma \right] \]

\[ R \sim N\left[ -\frac{\pi_1 - \alpha_1}{\pi_2 - \alpha_2} - \frac{\alpha_2}{\pi_2 - \alpha_2} \mu_z^\gamma, \frac{(-\alpha_2)^2}{(\pi_2 - \alpha_2)^2} \gamma \right] \]

Equating means and variances in (9) and (10) yields:

\[ [\alpha_2 + \theta] \pi_1 + [\theta \mu_x^\beta + \alpha_2 \mu_z^\gamma - \alpha_1] \pi_2 = \alpha_1 \theta + \theta \alpha_2 \mu_x^\beta - \theta \alpha_2 \mu_z^\gamma \]

\[ [\theta (\beta \gamma) + \alpha_2 (\gamma \gamma)] \pi_2 = \theta \alpha_2 [(\beta \gamma) - (\gamma \gamma)] \]

From these, the expressions for \( \pi_1 \) and \( \pi_2 \) in (6) are obtained. Q.E.D.

APPENDIX B

Derivation of the Correlation between Fees and Marginal Fees

PROPOSITION. Given the quadratic function \( f(z) = \pi_0 + \pi_1 z + \frac{1}{2} \pi_2 z^2 \) and the normally distributed random variable \( X \sim N(\mu, \sigma^2) \), the corre-
lation between \( f(X) \) and \( f'(X) \) is 
\[
\rho[f(X), f'(X)] = (1 + \frac{1}{2} CV^2[f'(X)])^{\frac{1}{2}},
\]
where \( CV(X) \) is the coefficient of variation of \( X \).

Proof. Given \( X \sim N(\mu, \sigma^2) \), we have 
\[
E(X^2) = \mu + \sigma^2, \quad E(X^4) = 3\mu^2 + \mu^4,
\]
and \( E(X^4) = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4 \). These moments imply that 
\( C(X, X^2) = 2\mu \sigma^2 \) and \( V(X^2) = 2\sigma^4 + 4\mu^2\sigma^2 \). Thus:
\[
C[f(X), f'(X)] = C[\pi_1 X + \frac{1}{2} \pi_2 X^2, \pi_2 X] = \pi_1 \pi_2 C(X, X) + \frac{1}{2} \pi_2 \sigma^2 C(X^2, X)
\]
\[
= (\pi_1 + \pi_2 \mu) \pi_2 \sigma^2,
\]
and:
\[
V[f'(X)] = \pi_2^2 \sigma^2,
\]
and:
\[
V[f(X)] = V[\pi_1 X + \frac{1}{2} \pi_2 X^2] = \pi_1^2 V(X) + \pi_1 \pi_2 C(X, X^2) + \frac{1}{4} \pi_2^2 V(X^2) = \sigma^2 [ (\pi_1 + \pi_2 \mu)^2 + \frac{1}{2} \pi_2^2 \sigma^2 ].
\]
Also:
\[
E[f'(X)] = \pi_1 + \pi_2 \mu
\]
and:
\[
CV[f'(X)] = V^{\frac{1}{2}}[f'(X)]/E[f'(X)] = (\pi_1 + \pi_2 \mu)/\pi_2 \sigma^2.
\]
Thus:
\[
\rho[f(X), f'(X)] = C[f(X), f'(X)]/(V[f(X)]xV[f'(X)])^{\frac{1}{2}} = (\pi_1 + \pi_2 \mu)\pi_2 \sigma^2/(\pi_2^2 \sigma^4 [ (\pi_1 + \pi_2 \mu)^2 + \frac{1}{2} \pi_2^2 \sigma^2 ] )^{\frac{1}{2}} = (1 + \frac{1}{2} \pi_2^2 \sigma^2/(\pi_1 + \pi_2 \mu)^2)^{-\frac{1}{2}} = (1 + \frac{1}{2} CV^2[f'(X)])^{-\frac{1}{2}}.
\]

REFERENCES


